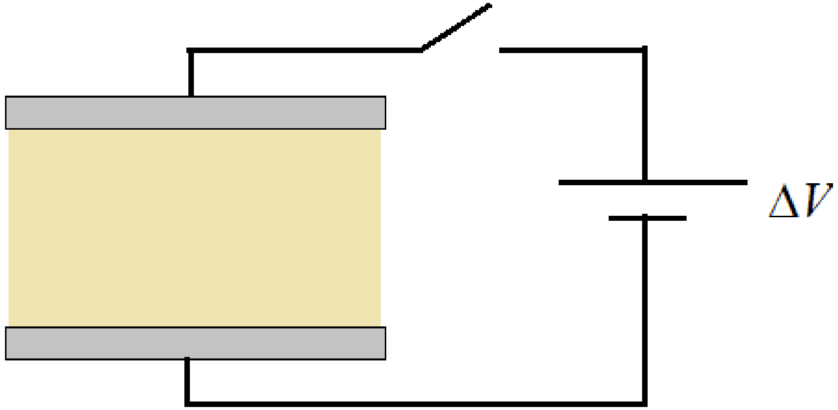
B.2 Capacitors

Capacitors are devices that store electric potential energy for later use. The disadvantage they have w/r to batteries is their relatively short lifespan – they have to be continually recharged. But an advantage is that multiple capacitors can be charged by a single battery, and these capacitors can then be combined to deliver low voltage power for a relatively long time (less time than the battery though), or combined differently to deliver a lot of power at a much higher voltage than the battery could do itself.

Most capacitors take a cylindrical shape, which we'll look at later, but the archetypal parallel-plate capacitor is easiest to analyze, and doesn't sacrifice any relevant details. They consist of two metal plates, usually filled with a dielectric in between. And then it's connected to a battery which will charge it.

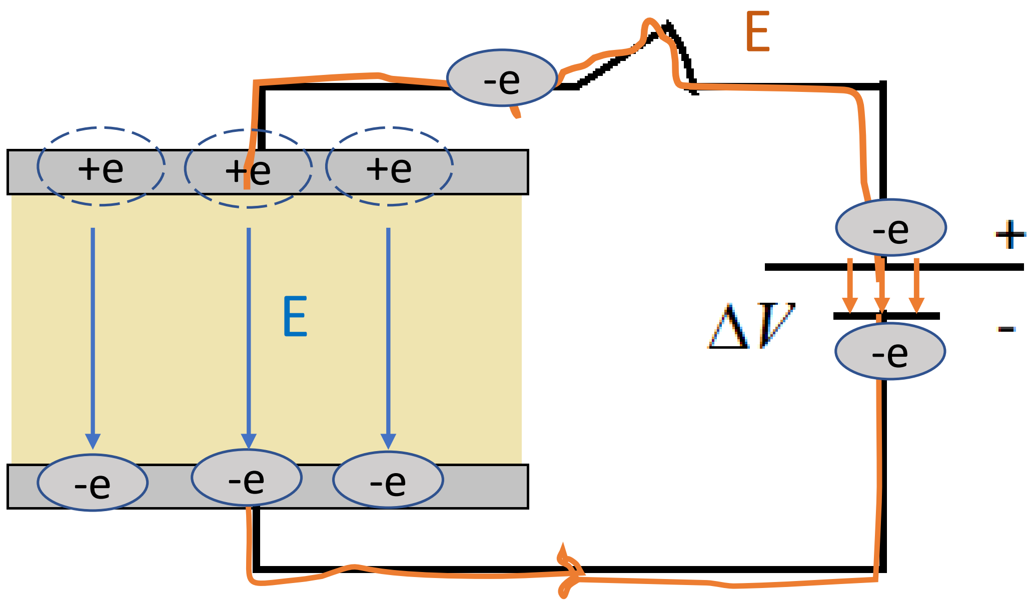


B.2 Capacitors

1. Battery's electric field draws electron from top plate to positive terminal, leaving a positively charged 'hole'.

4. An electric field line across the plates is generated.

5. Process will continue until potential difference across the capacitor equals that of the battery.



2. Battery's chemical reactions transports electron across terminals, against its internal electric field (force).

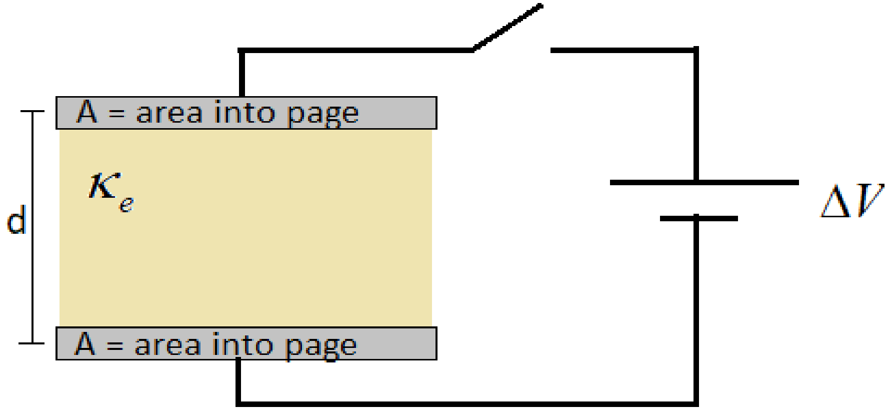
3. Battery's electric field pushes electron away from negative terminal onto bottom plate.

B.2 Capacitors

Of interest, is how much charge, Q, can be deposited on it before its potential difference equals the battery's.
The amount of charge depends on both the battery and the capacitor.

The relevant feature of the battery is its potential difference, ΔV.
Does Q go up or down with increasing ΔV?

up, because would need to put more charge on plate to get its potential difference up to ΔV.



The relevant feature of the capacitor is its dimensions and the dielectric its filled with. These factors are all combined into a single term called the ***capacitance*, C**. Would the capacitance go up or down if increase κ_e?, increase A?, increase d?

up, because it would reduce the field between plates and would require more charge to even its potential difference with the battery's.

up, because it would reduce the field too (by reducing σ)

down, because increasing d would increase the potential difference between the plates (ΔV~ Ed)

The relationship between Q, C, and ΔV is this: $Q = C\Delta V$

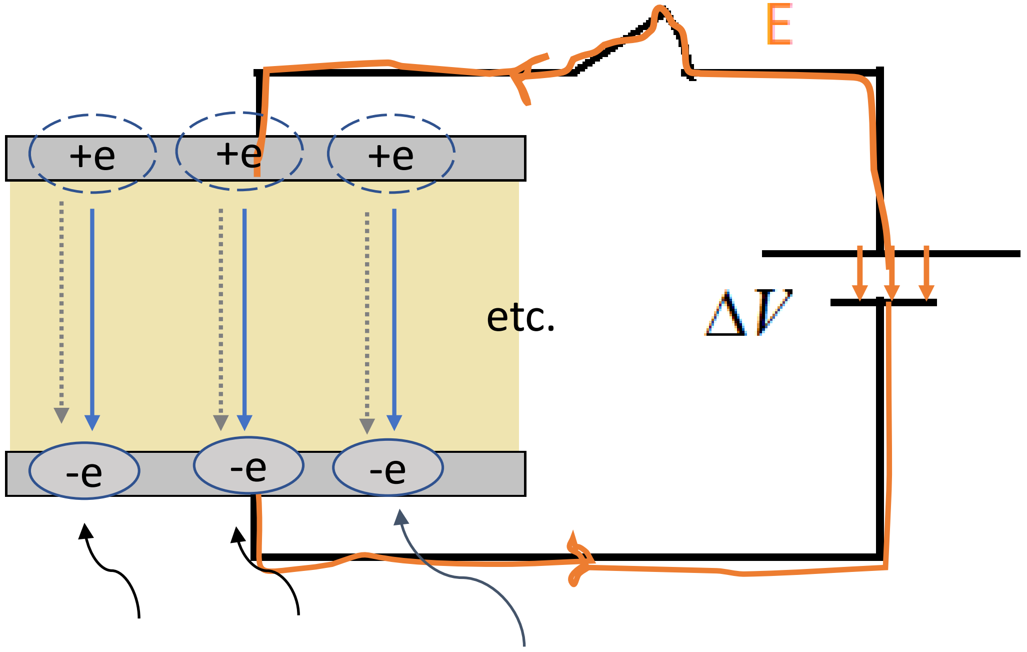
from the equation, we can see that the units of capacitance are: $\frac{\text{Coulombs}}{\text{Volt}} = \text{Farad (F)}$

B.2 Capacitors

Another thing we're interested in working out is the energy stored in the capacitor. We do already know what this is: it's the energy stored in the electric field between the plates, plus the energy stored in the stretched atoms in the dielectric. We previously found this to be:

$$\boxed{PE_{total} = \kappa_e \int \frac{E^2}{2\varepsilon_0} dV}$$

But there is another way to write this, which is more convenient for working with capacitors. The equivalent expression's derivation is as follows….



$$W_1 = e \cdot \Delta V_0 = e \cdot 0$$

$$W_2 = e \cdot \Delta V_1 = e \cdot \frac{e}{C}$$

$$W_3 = e \cdot \Delta V_2 = e \cdot \frac{2e}{C}$$

$$PE_{total} = \text{work battery } \textit{has} \text{ to do to charge the capacitor}$$

$$= W_{\text{charge}1} + W_{\text{charge}2} + W_{\text{charge}3} + \ldots + W_{\text{last charge}}$$

$$= 0 + e \cdot \frac{e}{C} + e \cdot \frac{2e}{C} + e \cdot \frac{3e}{C} + \ldots + e \cdot \frac{q}{C} + \ldots + e\frac{Q}{C}$$

$$= \int_0^Q dq \cdot \frac{q}{C}$$

$$= \frac{1}{2}\frac{q^2}{C}\Bigg|_{q=0}^{q=Q} = \frac{Q^2}{2C}$$

So that's it! On the formula sheet. I wrote it a bit differently. If you substitute: Q = CΔV into this expression, then you'd get:

$$PE_{total} = \frac{1}{2}C(\Delta V)^2$$

B.2 Capacitors

Now we want a formula for the capacitance of our *parallel plate capacitor*….the general procedure is to assume there is charge Q on it, work out what the potential difference ΔV must be in terms of Q, and then solve for C.

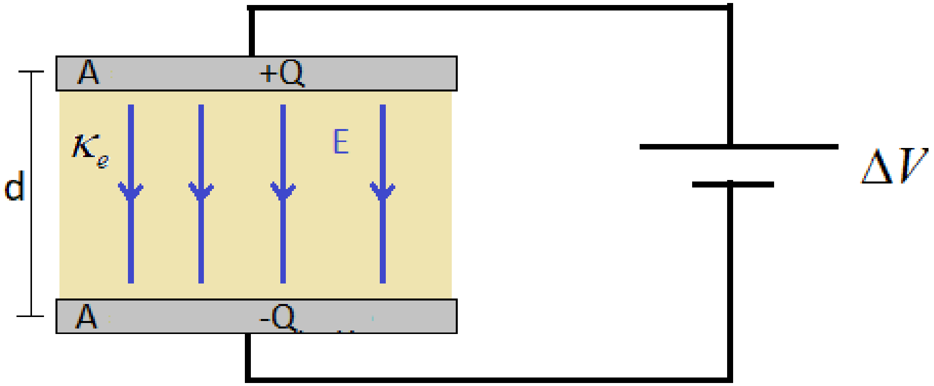
$$Q = C\Delta V$$

$$\Delta V = -\int_{bottom}^{top} \mathbf{E}\cdot d\mathbf{r} \;= Ed$$

$$= \frac{E_0}{\kappa_e} d$$

$$= \frac{1}{\kappa_e}\left(2\times\frac{\sigma}{2\varepsilon_0}\right) d$$

$$= \frac{1}{\kappa_e}\left(\frac{Q/A}{\varepsilon_0}\right) d$$

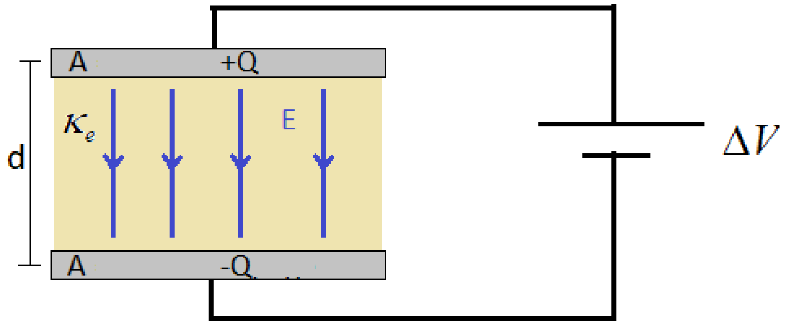


$$Q = C\cdot\frac{1}{\kappa_e}\left(\frac{Q/A}{\varepsilon_0}\right) d \longrightarrow 1 = C\cdot\frac{1}{\kappa_e}\frac{d}{A\varepsilon_0} \longrightarrow \boxed{C = \kappa_e \frac{A\varepsilon_0}{d}}$$

Other capacitor shapes (cylindrical, spherical, etc., have slightly different formulas)

B.2 Capacitors

So say we have a parallel plate capacitor consisting of two *10cm×10cm plates*, a distance 4cm apart, filled with pyrex ($\kappa_e = 6$). And that we hook it up to a 120V battery.



(a) What is its capacitance?

$$C = \kappa_e \frac{A\varepsilon_0}{d} = (6)\frac{(0.10)^2(8.85\times 10^{-12})}{(0.04)} = 1.3\times 10^{-11}\,\mathrm{F} = 13\,\mathrm{pF}$$

(b) How much charge is stored on it?

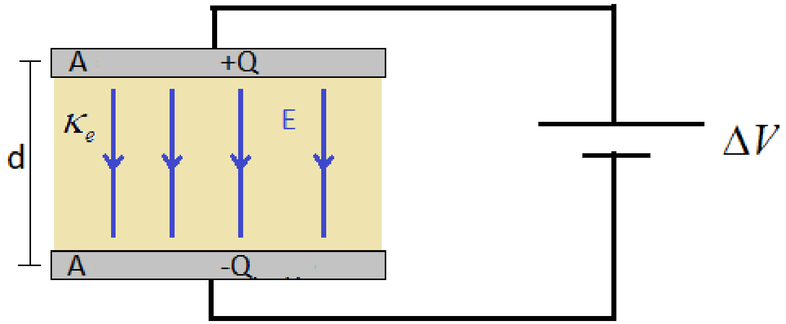
$$Q = C\Delta V = (13\mathrm{pF})(120\mathrm{V}) = 1560\,\mathrm{pC} = 1.56\,\mathrm{nC}$$

(c) How much energy does it store?

$$PE_{total} = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}(13\times 10^{-12})(120)^2 = 1.9\times 10^{-7}\,\mathrm{J} = 190\,\mathrm{nJ}$$

B.2 Capacitors

(d) Pyrex has a dielectric strength of 14MN/C. What potential difference would have to be applied to the capacitor to reach this strength?



$$\Delta V = -\int_{bottom}^{top} \mathbf{E} \cdot d\mathbf{r}$$

$$\Delta V = Ed$$

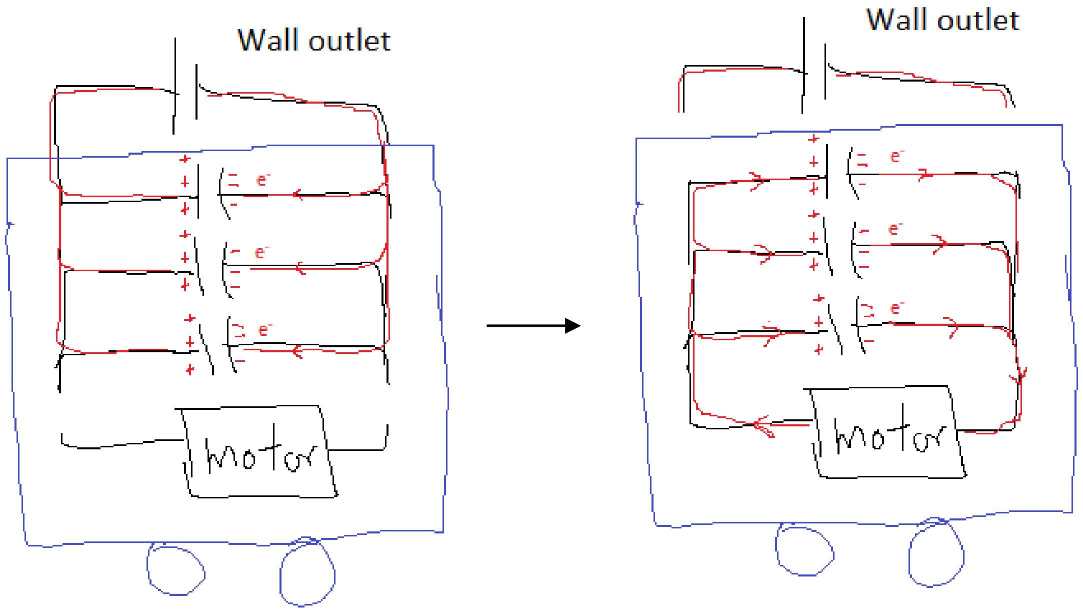
$$\Delta V = (14 \times 10^6)(0.04) = 5.6 \times 10^5\,\mathrm{V} = 560\,\mathrm{kV}$$

(e) And what would be the energy then?

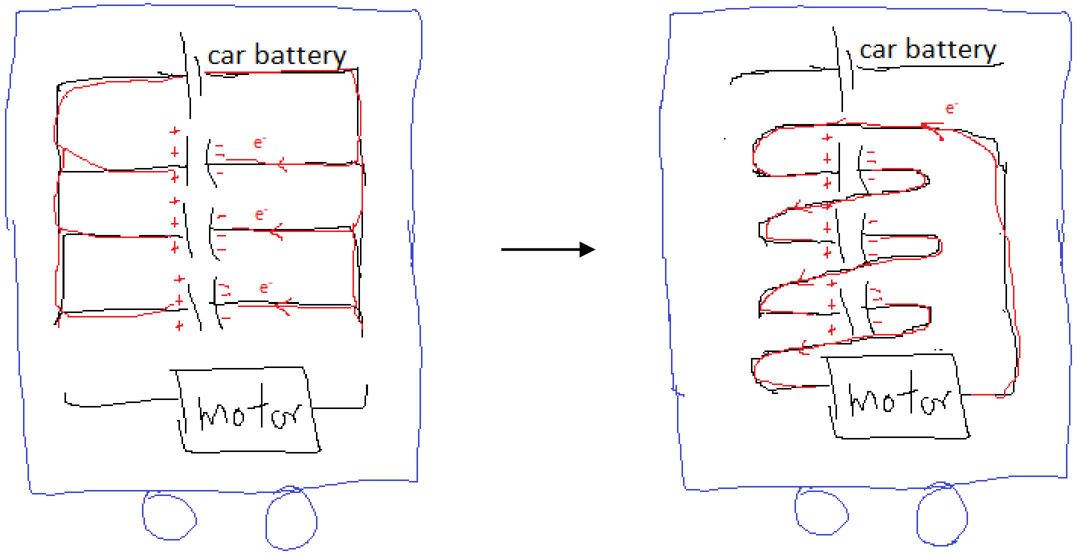
$$PE_{total} = \frac{1}{2} C (\Delta V)^2 \quad = \frac{1}{2}(13 \times 10^{-12})(5.6 \times 10^5)^2 \quad = 4.1\,\mathrm{J}$$

B.2 Capacitors

Now we want to consider combinations of capacitors. For instance, consider the following scenarios….



Then disconnect capacitor bank from outlet and connect to motor. Capacitors discharge slowly, at same potential as outlet, delivering steady power to motor.



Car battery charges bank of capacitors in electric car.

Wall outlet charges bank of capacitors in go-cart.

Then disconnect capacitor bank from battery and connect to motor in different fashion. Now capacitors discharge quickly, with super high potential, delivering a lot of power motor.

B.2 Capacitors

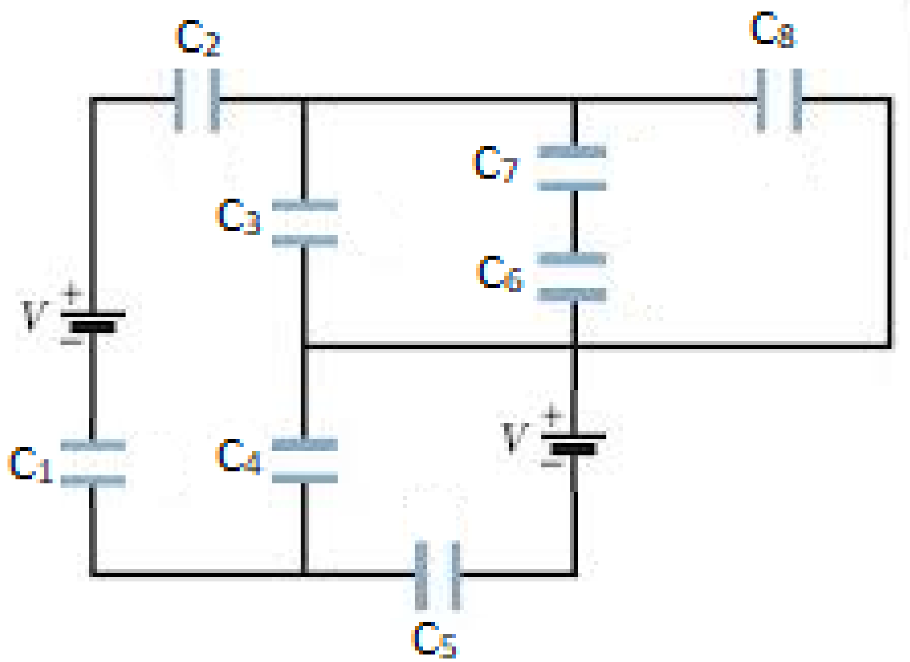
Ultimately we want to know a few things about any general network of capacitors.

(1) Charge and energy stored in each capacitor
(2) Charge and energy that would be delivered to a ‘motor’ if the bank of capacitors were discharged through it.

There are two methods to address these question:

(a) Method of equivalent capacitance.
(b) Kirchoff’s laws.

The first is faster, when applicable, and the latter more general. We’ll consider the first, first.



B.2 Capacitors

Method of equivalent capacitance aims to simplify a network of capacitors by systematically replacing certain combinations of capacitors with a single 'equivalent' capacitor. The two capacitor combinations amenable to this method are *series*, and *parallel*. Two capacitors are connected 'in series' or 'in parallel' if they satisfy the following criteria:

Series: Can draw a path along circuit, from one capacitor to the other, w/o crossing a junction.

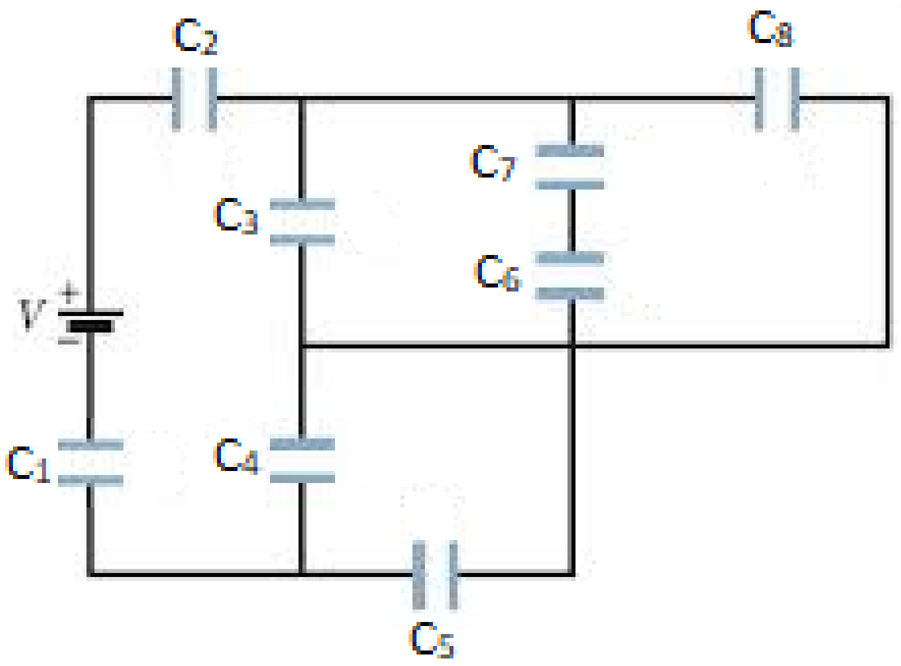
Parallel: Can draw a closed loop along the circuit, from one capacitor to the other, w/o crossing another circuit element (i.e., battery, capacitor, resistor, etc.)

Which capacitors are in parallel?

C_4 and C_5
C_3 and C_8

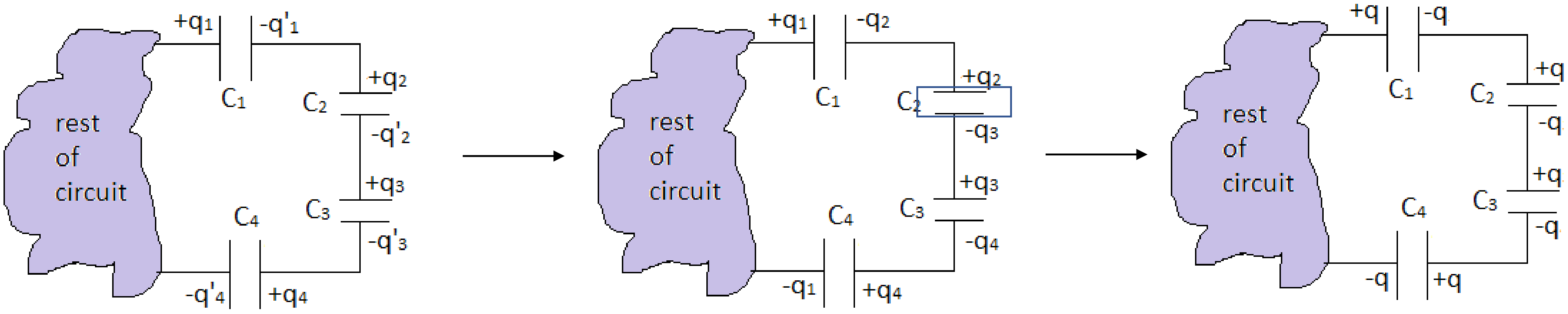
Which capacitors are in series?

C_1 and C_2
C_6 and C_7



B.2 Capacitors

Fun Fact: capacitors in series all carry the same charge



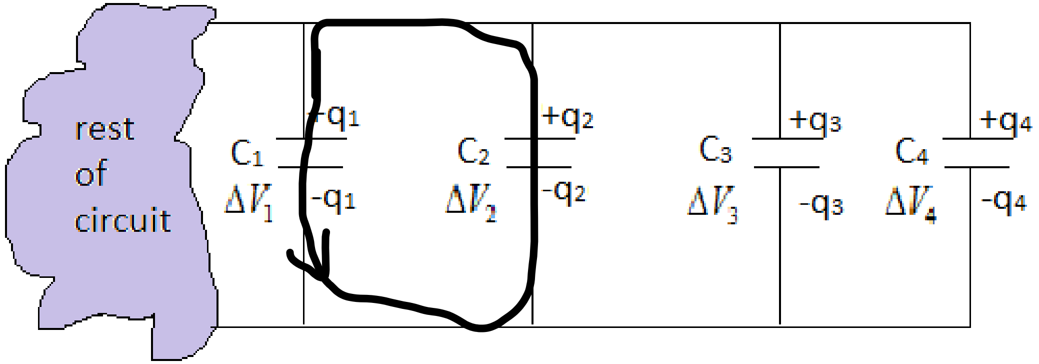
Suppose they were different….

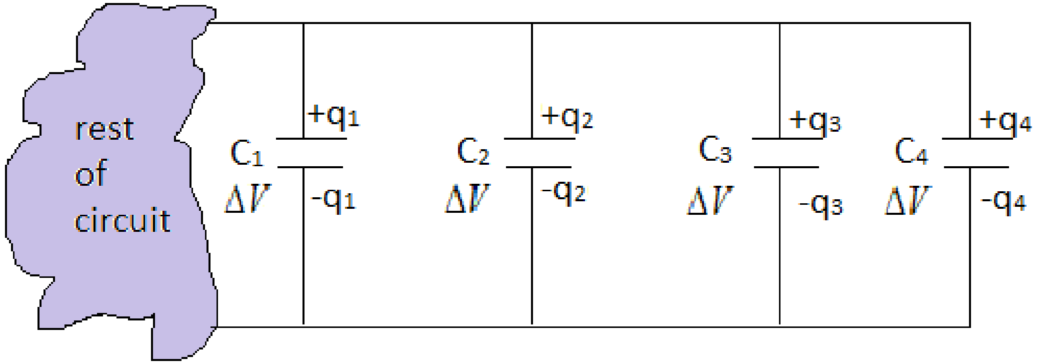
q_1 must equal q'_4 due to charge conservation – an overall segment of wire must be neutral. Or in other words, whatever charge flows out of the segment, must flow back in. For same reason q'_1 must equal q_2, and q'_2 must equal q_3, and q'_3 must equal q_4.

Now draw Gaussian surface around capacitor. Field in metal must be zero, and so flux must be zero. But then enclosed charge must be zero. So $q_2 = q_3 = q$. And similarly for the others.

B.2 Capacitors

Fun Fact: capacitors in parallel all carry the same voltage difference





Just draw a loop and see that it cannot be otherwise:

$$\Delta V_{closed\ loop} = 0$$

$$-\Delta V_1 + \Delta V_2 = 0$$

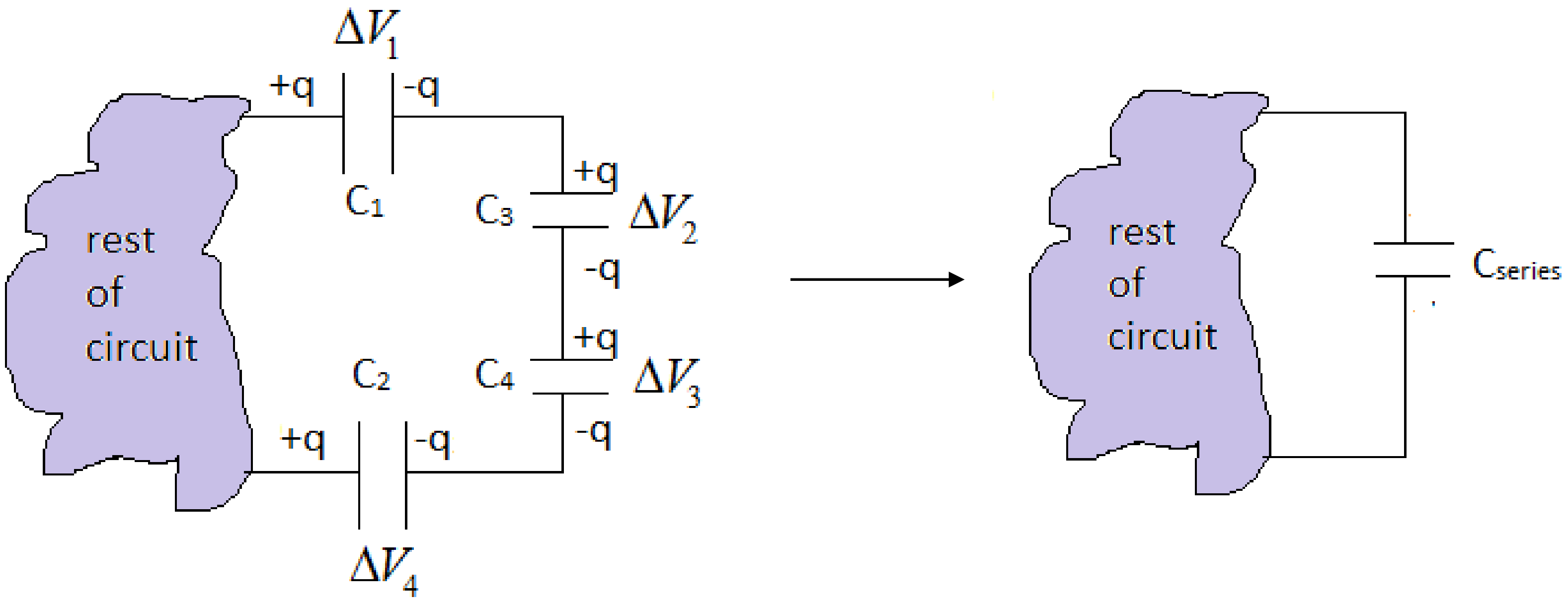
$$\Delta V_1 = \Delta V_2$$

And same for others.

B.2 Capacitors

Series equivalent capacitance

Now we need to figure out what the equivalent capacitance of a bunch of capacitors connected in series, is. The *equivalent capacitance* is defined as the single capacitor that would draw the same charge under same voltage, as the series combination.



$$\Delta V_{series} = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4$$

$$q_{series} = q$$

$$C_{series} = \frac{q_{series}}{\Delta V_{series}} = \frac{q}{\Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4}$$

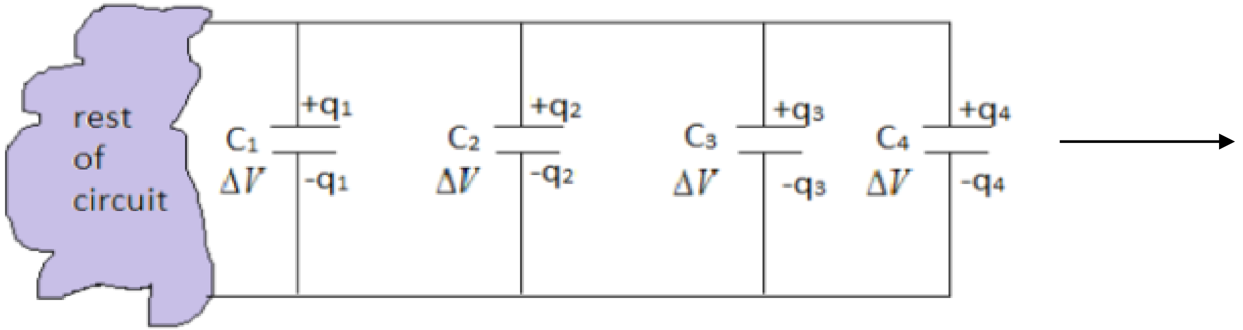
$$= \frac{1}{\Delta V_1 / q + \Delta V_2 / q + \Delta V_3 / q + \Delta V_4 / q} = \frac{1}{C_1^{-1} + C_2^{-1} + C_3^{-1} + C_4^{-1}}$$

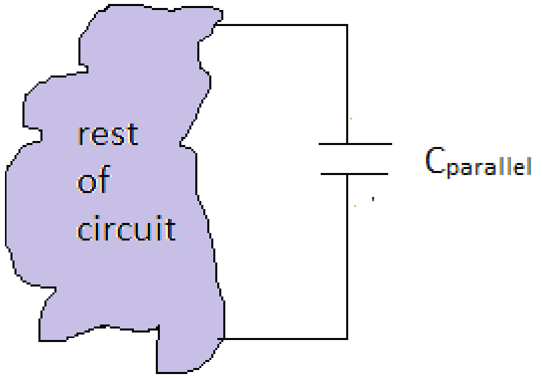
$$C_{series} = (C_1^{-1} + C_2^{-1} + C_3^{-1} + ... + C_n^{-1})^{-1}$$

B.2 Capacitors

Parllel equivalent capacitance

Now we need to figure out what the equivalent capacitance of a bunch of capacitors connected in series, is. The *equivalent capacitance* is defined as the single capacitor that would draw the same charge under same voltage, as the series combination.





$$\Delta V_{parallel} = \Delta V$$

$$q_{parallel} = q_1 + q_2 + q_3 + q_4$$

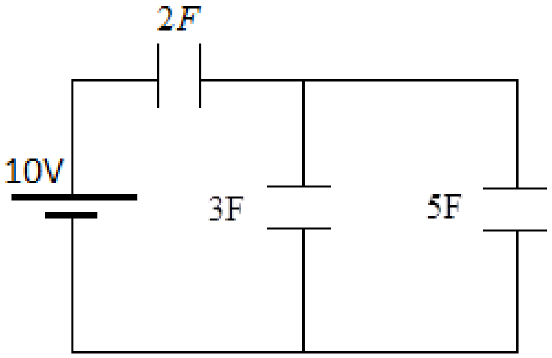
$$C_{parallel} = \frac{q_{parallel}}{\Delta V_{parallel}} = \frac{q_1 + q_2 + q_3 + q_4}{\Delta V}$$

$$= \frac{q_1}{\Delta V} + \frac{q_2}{\Delta V} + \frac{q_3}{\Delta V} + \frac{q_4}{\Delta V} = C_1 + C_2 + C_3 + C_4$$

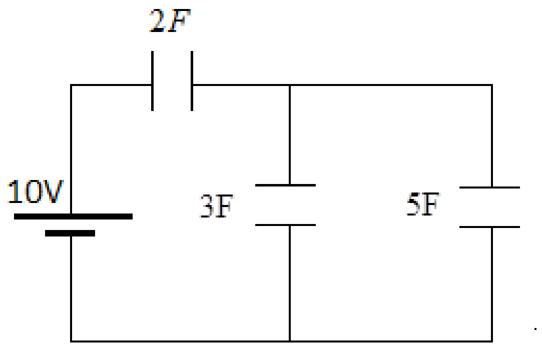
$$C_{parallel} = C_1 + C_2 + C_3 + ... + C_n$$

B.2 Capacitors

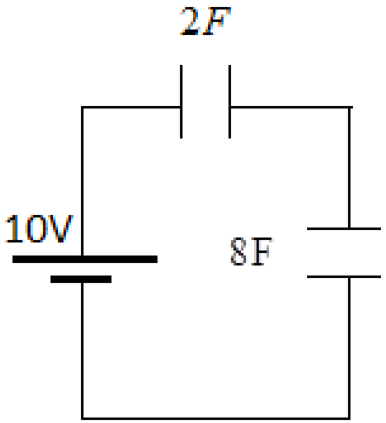
Consider the following amazing circuit. What's the charge on each capacitor? If the battery were disconnected and some device inserted between the terminals, how much charge and energy would flow through the device?



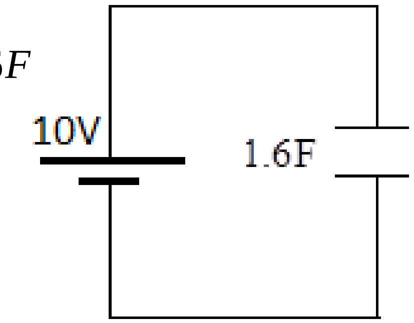
General procedure is to ultimately reduce it to one capacitor.
We get:



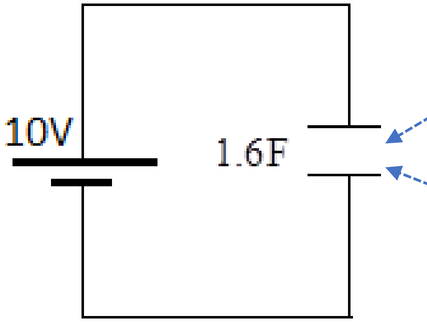
$$C_{parallel} = 3F + 5F = 8F$$



$$C_{series} = \left(\frac{1}{2F} + \frac{1}{8F}\right)^{-1} = 1.6F$$



B.2 Capacitors



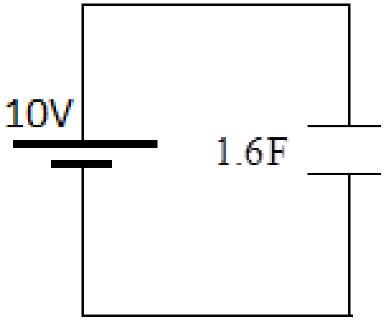
$$q = C\Delta V = (1.6\text{F})(10\text{V}) = 16\text{C}$$

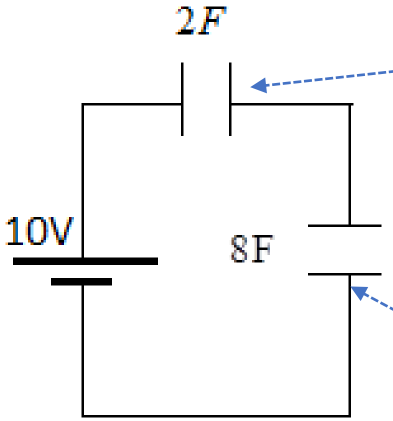
This is the charge that would flow through a device connected to the capacitors.

$$PE = \frac{1}{2}C\Delta V^2 = \frac{1}{2}(1.6)(10)^2 = 80\text{J}$$

This is the energy that would flow through the device.

And now let's work backwards to get the charges on the individual capacitors…



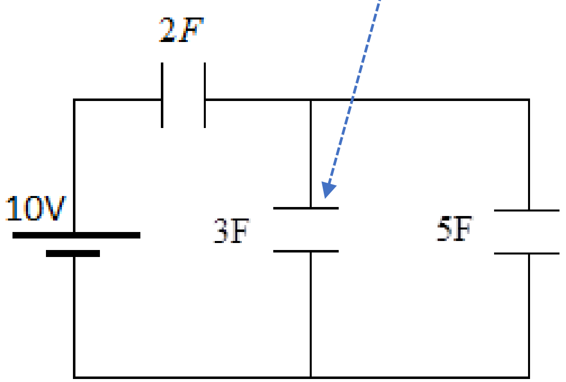


$$q = 16\text{C}$$
$$\Delta V = \frac{q}{C} = \frac{16}{2} = 8\text{V}$$

$$q = 16\text{C}$$
$$\Delta V = \frac{q}{C} = \frac{16}{8} = 2\text{V}$$

$$q = 16\text{C}$$
$$\Delta V = 8\text{V}$$

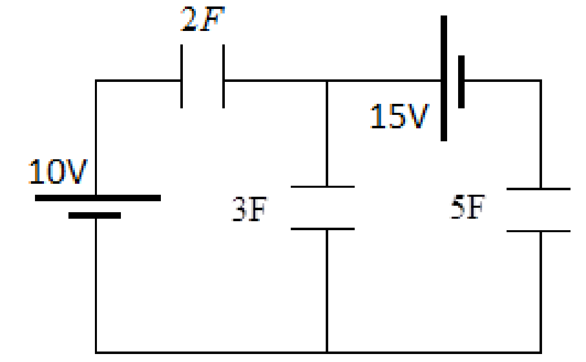
$$\Delta V = 2\text{V}$$
$$q = C\Delta V = (3\text{F})(2\text{V}) = 6\text{V}$$



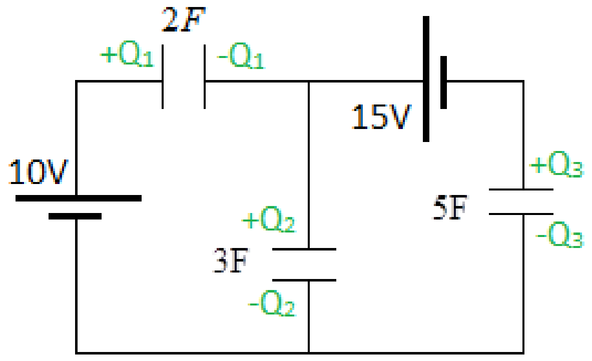
$$\Delta V = 2\text{V}$$
$$q = C\Delta V = (5\text{F})(2\text{V}) = 10\text{C}$$

B.2 Capacitors

Just when you thought the circuit couldn't get any more amazing, this happened.   Now how much energy is stored in the circuit? What are the charges on the capacitors?

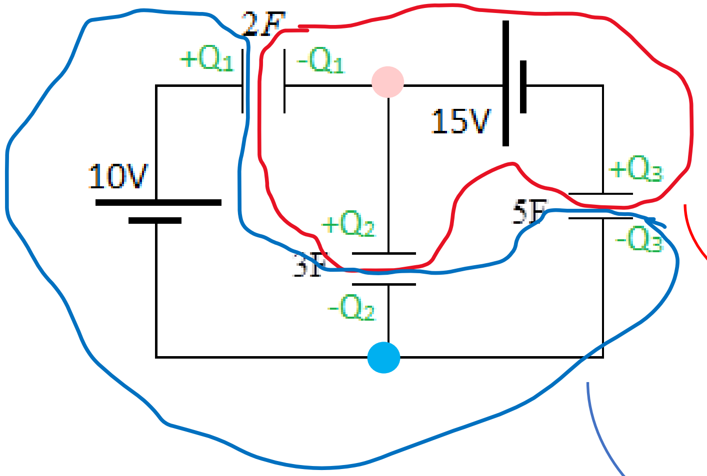


In this case we cannot combine any of the capacitors in series or parallel. So we must take another route. This is how it works.



1. Label the charges in the wires in any arbitrary way:

B.2 Capacitors



2. At each junction, apply KCL (Kirchoff's charge law), which states that the sum of the charges leaving a junction must add up to zero.

$$\sum Q's = 0 \qquad KCL$$

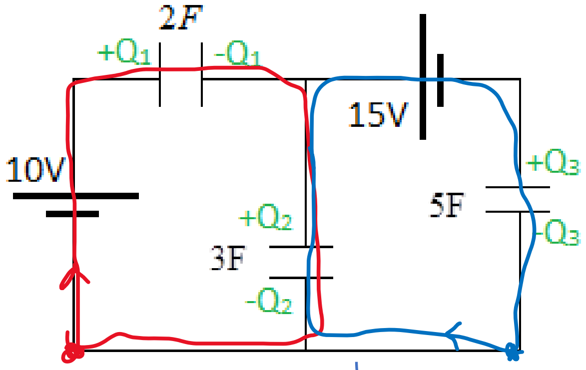
$$-Q_1 + Q_2 + Q_3 = 0$$

$$Q_1 - Q_2 - Q_3 = 0$$

Note we needn't apply it to the other junction because we'll get the same equation.

$$Q_1 - Q_2 - Q_3 = 0$$

B.2 Capacitors



3. Apply KVL (Kirchoff's Voltage Law) to as many loops as necessary (# of unknown charges -1), which states that the sum of the potential differences around a closed loop must add up to zero.

$$\sum \Delta V's = 0 \qquad KVL$$

The loops you choose, their starting point, and the direction in which you circumambulate them are all arbitrary.

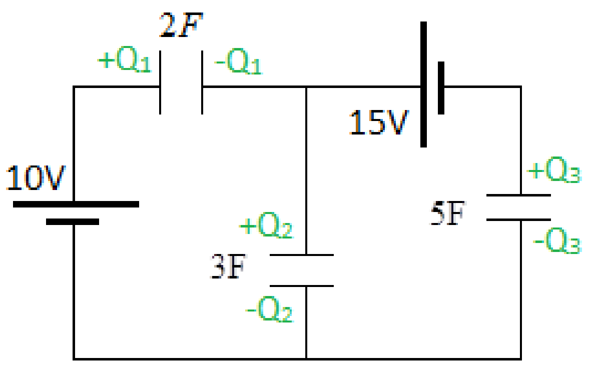
$$+10 - \frac{Q_1}{2} - \frac{Q_2}{3} = 0$$

$$\frac{Q_1}{2} + \frac{Q_2}{3} = 10$$

$$\frac{Q_2}{3} - 15 - \frac{Q_3}{5} = 0$$

$$\frac{Q_2}{3} - \frac{Q_3}{5} = 15$$

B.2 Capacitors



4. Solve the KCL, KVL equations for the Q's

These are the three equations so far:

$$Q_1 - Q_2 - Q_3 = 0$$

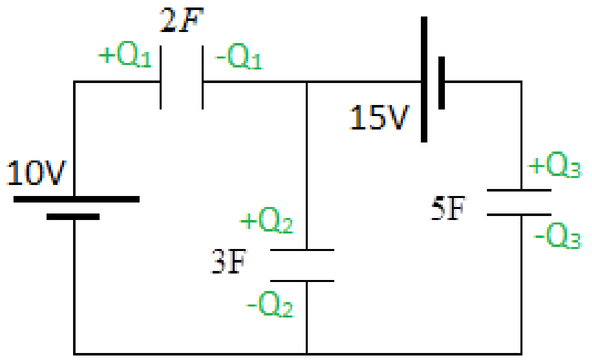
$$\frac{Q_1}{2} + \frac{Q_2}{3} = 10$$

$$\frac{Q_2}{3} - \frac{Q_3}{5} = 15$$

You can use a matrix to solve them if you wish. It would look like this:

$$\begin{pmatrix} 1 & -1 & -1 \\ 1/2 & 1/3 & 0 \\ 0 & 1/3 & -1/5 \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix} \longrightarrow \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1/2 & 1/3 & 0 \\ 0 & 1/3 & -1/5 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix} \longrightarrow \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 28.5 \\ -27.5 \end{pmatrix}$$

B.2 Capacitors



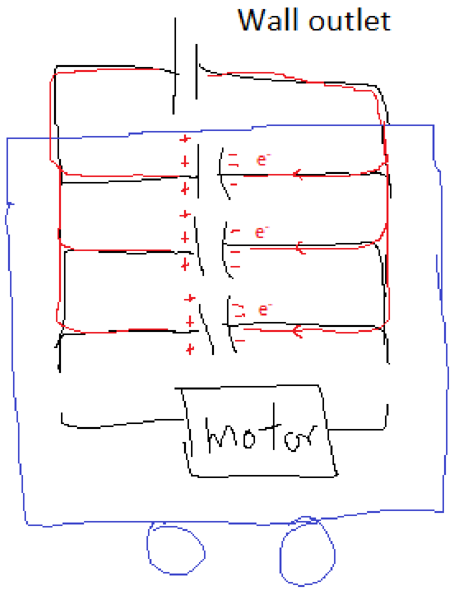
$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 28.5 \\ -27.5 \end{pmatrix}$$

The total energy stored would be:

$$PE = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} + \frac{Q_3^2}{2C_3}$$

$$= \frac{1^2}{2(2)} + \frac{28.5^2}{2(3)} + \frac{(-27.5)^2}{2(5)}$$

$$= 211\text{J}$$


Wall outlet
e⁻
e⁻
e⁻
motor